Analysis of Algorithms

Part 1

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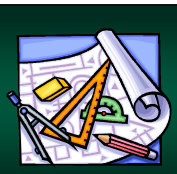

Analysis of Algorithms

Looking at how well it works

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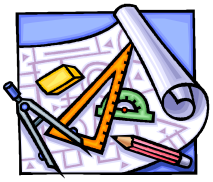
What is an Algorithm?

- An *algorithm* is a sequence of unambiguous instructions that solves a problem
- Can be represented various forms – i.e. languages
- Each unique set of data fed into an algorithm specifies an *instance* of that algorithm



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Analysis of Algorithms

- Algorithms must to analyzed to determine whether it should be used
- This field is called *algorithmics*
- How it is analyzed:
  - correctness
  - unambiguity
  - effectiveness
  - finiteness/termination - does it in a *finite* amount of time

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Correctness

- Correctness means the algorithm obtains the required output with *valid* input
- In other words, does it do what it is supposed to do
- *Proof of Correctness* can be easy for some algorithms – and quite difficult for others
- *Proof of Incorrectness* is quiet easy – find one instance where it fails on valid input

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Effectiveness

- How good is the algorithm?
- Two major areas of interest:
  - *time efficiency* defines how long the algorithm will take to complete
  - *space efficiency* defines how much memory and resources will be needed
- … and how these react as the data set grows

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Effectiveness

- Knowing this, we can determine if there is a better algorithm
- Does there exist a better algorithm?
  - better time complexity
  - better space efficiency
- *Efficiency is a **HUGE** part of creating professional programs*

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Determining Effectiveness

Is the algorithm good?

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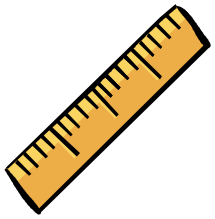
Determining Effectiveness

- Determining if an algorithm is efficient – and will work best to solve a problem – *is vital*
- Some algorithms may work incredibly well
- … and sometimes fail horribly



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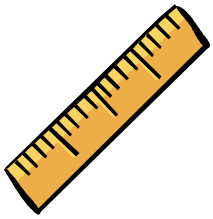
Determining Effectiveness

- Moreover, some algorithms are sensitive to the type of data
- And two algorithms – which solve the same problem – may act *differently* given a set of values



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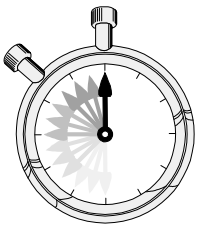
Time Complexity

- Given that computer programs are designed to be fast (and thus efficient), estimating how long an algorithm will take is useful
- *What is the task, repeated by the algorithm, has the most impact on the time?*



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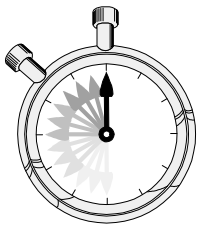
Time Complexity

- The *basic operation* contributes the ***most*** towards the running time of the algorithm
- It is the task that is repeated (often in a loop) by the algorithm



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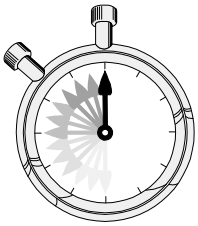
Time Complexity

- We could time the basic operation and then estimate how many times its executed
- This will give a rough idea of the total runtime



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Theoretical Analysis of Time Efficiency

Total execution time

Number of times the basic operation executes

$$T(n) \approx c_{op}C(n)$$

number of items in the data set

Cost: execution time of a basic operation

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Empirical analysis of time efficiency

- *Empirical Analysis* can be performed by *observation*
- Select a specific (typical) sample of inputs
- Then physical unit of time and / or count actual number of basic operation's executions
- Analyze the data to estimate: $T(n)$, C_{op}, and $C(n)$

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Time Complexity Cases

- Worst case: $C_{worst}(n)$
  - maximum executions over a set of size n
  - can be linear, quadratic, or even exponential!
  - the worst case can be exceedingly rare
- Best case: $C_{best}(n)$
  - minimum executions over a set of size n
  - best case can also be exceedingly rare

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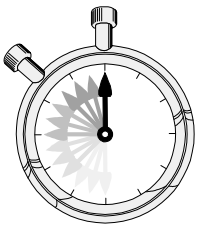
Time Complexity Cases

- Average case: $C_{avg}(n)$
  - execute with typical data…
  - …in other words, the type of data the algorithm will it normally encounter
  - this is **NOT** the average of worst and best case



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Order of Growth

Uh, "O"

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Order of Growth

- For some algorithms, efficiency depends on the *form* of input
- Sometimes, the order of data, or the type of data can *drastically* increase cost

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Order of Growth

- In the previous equation, notice that C(n) represents the total number of times the basic operation is executed
- But, how does C react to n?

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Order of Growth

- We can analyze n as it approaches ∞
- Examples:
  - how will it run on a computer that is twice as fast?
  - how long does it take with twice the input?

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Order of Growth

- One property of functions that we are interested in its rate of growth
- *Rate of growth* doesn't simply mean the "slope" of the line associated with a function
- Instead, it is more like the *curvature* of the line

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Several Growth Functions

- There are several functions
- In increasing order of growth, they are:
  - Constant ≈ 1
  - Logarithmic ≈ log n
  - Linear ≈ n
  - Log Linear ≈ n log n
  - Quadratic ≈ n^2
  - Exponential ≈ 2^n

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Growth Rates Compared

n =	1	2	4	8	16
1	1	1	1	1	1
log n	0	1	2	3	4
n	1	2	4	8	16
n log n	0	2	8	24	64
n^2	1	4	16	64	256
n^3	1	8	64	512	4096
2^n	2	4	16	256	65536

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Classifications

- Using the known growth rates…
  - algorithms are classified using three notations
  - these allows you to see, quickly, the advantages/disadvantages of an algorithm
- Major notations:
  - Big-O
  - Big-Theta
  - Big-Omega

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Order of Growth

Notation	Name	Meaning
O (*n*)	Big-O	class of functions f(n) that grow no faster than n
Θ (*n*)	Big-Theta	class of functions f(n) that grow at same rate as n
Ω (*n*)	Big-Omega	class of functions f(n) that grow at least as fast as n

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Big-O

- So, Big-O notation gives an *upper bound* on growth of an algorithm
- We will use Big-O almost exclusively rather than the other two

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Big-O

- The following means that the growth rate of f(n) is no more than the growth rate of n
- This is one of the classifications mentioned earlier

$$f(n) \text{ is } O(n)$$

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Why it is O-some!

- These classes make it is easy to…
  - compare algorithms for efficiency
  - making decisions on which algorithm to use
  - determining the scalability of an algorithm
- So, if two algorithms are the same class…
  - they have the same rate of growth
  - both are equally valid solutions

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O(1)

- Represents a constant algorithm
- It does not increase / decrease depending on the size of n
- Examples
  - appending to a linked list *(with an end pointer)*
  - array element access
  - practically all simple statements

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O(log n)

- Represents logarithmic growth
- These increase with n, but the rate of growth diminishes
- For example: for base 2 logs, the growth only increases by one each time n doubles

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O(log n) Examples

- Searching for an item on a sorted array – (e.g. a binary search)
- Traversing a sorted tree

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O(n)

- Represents an algorithm that grows linearly with n
- Very common in programming – for iteration
- Examples:
  - finding an item in a linked list
  - merging two sorted arrays

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O(n log n)

- Represents an algorithm that has "log linear" growth
- These algorithms grow based on both n and n's log value

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O(n log n) Examples

- Quick Sort
- Heap Sort
- Merge Sort
- Fourier transformation

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O(n^2)

- Represents an algorithm that has "quadratic" growth
- These algorithms grow dramatically fast depending on the size of n
- **Do NOT use for large values of n**

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$O(n^2)$ Examples

- Bubble Sort, Selection Sort, etc….
- matrix multiplication
- merging unsorted arrays

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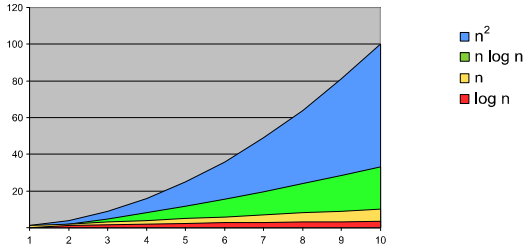
Growth: 1 to 10



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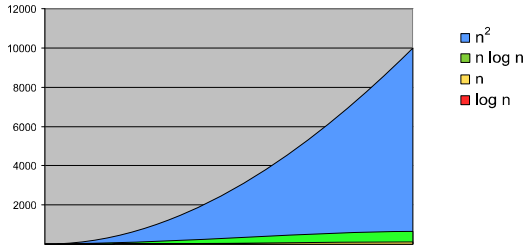
Growth: 1 to 100



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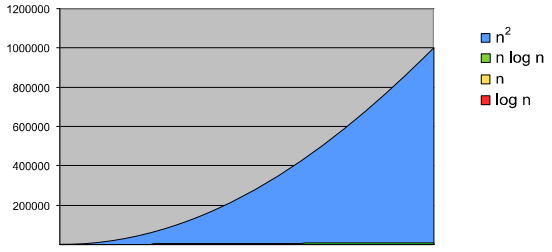
Growth: 1 to 1000



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Seconds needed (1 Microsecond Operation)

n	O(log n)	O(n)	O(n log n)	$O(n^2)$
10	0.000003	0.000010	0.000033	0.000100
100	0.000007	0.000100	0.000664	0.010000
1,000	0.000010	0.001000	0.009966	1.000000
10,000	0.000013	0.010000	0.132877	100.000000
100,000	0.000017	0.100000	1.660964	2.8 hours
1,000,000	0.000020	1.000000	19.931569	11.6 days
10,000,000	0.000023	10.000000	3.9 minutes	3.16 years
100,000,000	0.000027	100.000000	44.3 minutes	316.9 years
1,000,000,000	0.000030	17 minutes	8.3 hours	31,689 years

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Big-O Summary

Good

Bad

O(1)	O(wow!)
O(log n)	O(yeah!)
O(n)	O(nice!)
O(n log n)	O(kay!)
$O(n^2)$	O(no!)
$O(2^n)$	O(sh*t!)

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Big-O Math

Time for a "Big-O" headache

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Asymptotic Analysis

- Any algorithm can be analyzed, and its complexity/growth can be written as a simple mathematical expression
- *Asymptotic analysis* of an algorithm determines the running time in big-O notation

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Asymptotic Analysis

1. Find the worst-case number of primitive operations executed as a function of the input size
2. Eliminate meaningless values once the base rate in found

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Example

- If we analyze an algorithm and find it executes **12 * n – 1**
  - constant factors and lower-order terms dropped since they become meaningless for large values of n
  - remember, this is a *growth rate*
- It will be **O(n)**

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Examples

$3000n + 7 \rightarrow O(n)$

$2n^5 + 3n^3 + 5 \rightarrow O(n^5)$

$7n^3 - 2n + 3 \rightarrow O(n^3)$

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Test Your Might…

```
for(i = 0; i < 100; i++)
{
   total += values[i];
}
```

O(1)

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Test Your Might…

```
for(x = 0; x < array.size; x++)
{
    sum += array[x];
}

for(x = 0; x < array.size; x++)
{
    sum -= array[x];
}
```

O(n)

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Test Your Might…

```
for (x = 0; x < array.size; x++)
{
   for (y = 0; y < x; y++)
   {
      sum += x - y;
   }
}
```

O(n^2)

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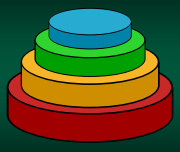

Towers of Hanoi

A Classic Stack Puzzle

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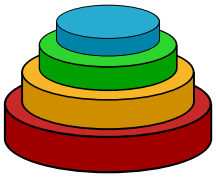
Towers of Hanoi

- *Towers of Hanoi* is a famous puzzle created by mathematician Edouard Lucas in 1883
- It is based on a "legend"



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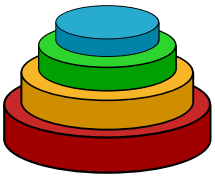
The Puzzle



- Consists of a collection of discs with unique diameters
- Each disc has a hole in the center used to place it on one of 3 different pegs

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The Puzzle

- Goal:
  - starts with all the discs stacked on one peg
  - goal is to move all the discs to another peg
- Gameplay:
  - a disc cannot be placed onto a smaller disc
  - only one disc can be moved at a time

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The Legend

- Well, the legend was created along with the puzzle and expanded over time
- Basically, somewhere in a *hidden place*, priests are moving a stack of 64 discs
- The ancient prophecy states that when the entire stack is moved…the World ***ENDS!***

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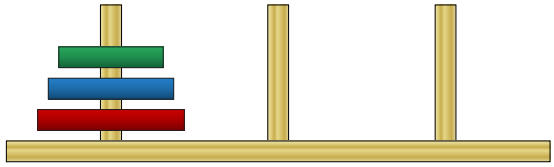
Hanoi: 3 Discs



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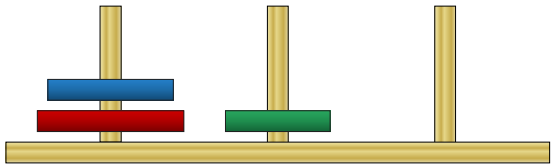
Hanoi: 3 Discs



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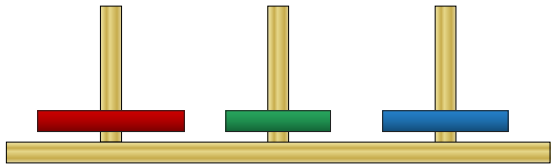
Hanoi: 3 Discs



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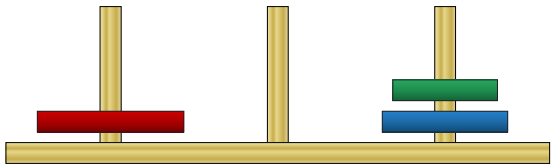
Hanoi: 3 Discs



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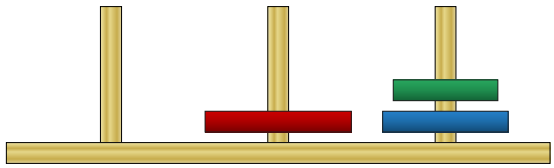
Hanoi: 3 Discs



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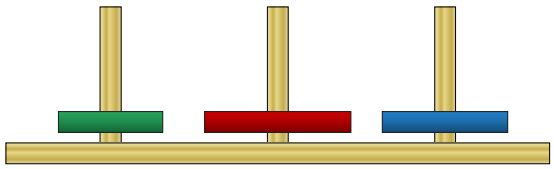
Hanoi: 3 Discs



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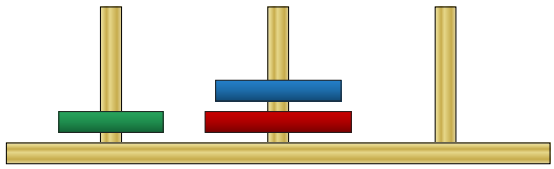
Hanoi: 3 Discs



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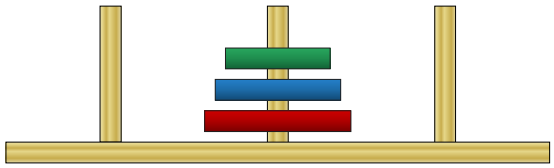
Hanoi: 3 Discs



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Hanoi: Solution

- An elegant solution is to use recursion
- Since disks are move from each tower using LIFO, each tower can be represented as a stack
- The "classic" recursive solution just shows what actions to take, it doesn't move any values… but you could modify it easily to.

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Hanoi: in Java

```
void hanoi(int disc, Stack from, Stack temp, Stack dest)
{
   if (disc == 1)
   {
      move(from, dest);  //base case
   }
   else
   {
      hanoi(disc - 1, from, dest, temp);
      move(from, dest);
      hanoi(disc - 1, temp, from, dest);
   }
}
```


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Hanoi: Demo Version

```
void hanoi(int disc, Stack from, Stack temp, Stack dest)
{
   if (disc == 1)
   {
      System.out.println(disc + ": " + from + " to " + dest);
   }
   else
   {
      hanoi(disc - 1, from, dest, temp);
      System.out.println(disc + ": " + from + " to " + dest);
      hanoi(disc - 1, temp, from, dest);
   }
}
```


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Hanoi: Demo Version

```
// Disc 1 is the *smallest* disc.
// We start recursion with the BIGGEST disc.

void main()
{
   hanoi(3, 'A', 'B', 'C');
}
```


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Hanoi: Demo Output

```
1: A to C
2: A to B
1: C to B
3: A to C
1: B to A
2: B to C
1: A to C
```


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Hanoi: Time Complexity

- The minimum number of moves required for a stack of N discs is is 2^N-1
- So, the time complexity of the Towers of Hanoi puzzle is $O(2^n)$ - exponential!

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Hanoi: Is the World Ending?

- The "legend" states that the monks have to move 64 discs… order of 2^{64}
- So…
  - if they take one second to move each disc, it will take them 584,542,046,090 years!
  - if a super-computer moves a disc once per microsecond, it still takes 584,542 years!

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